Jan 31, 2014

Today: Using the Derivative

Moral of Today: Curves are hard, lines are fun.

Newtons Method

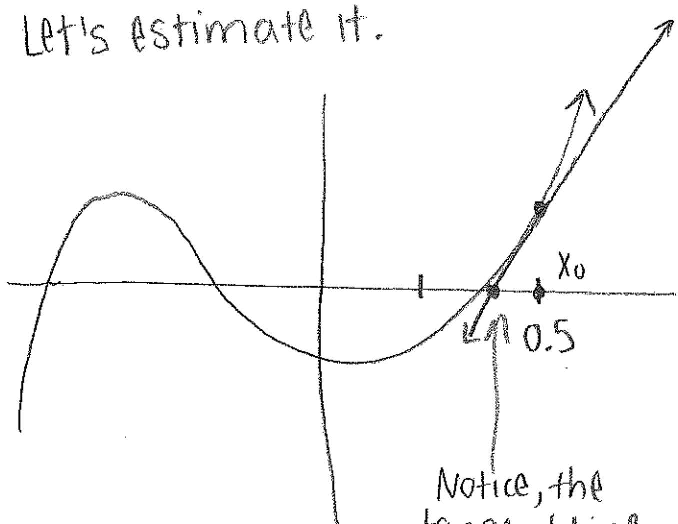
* helps us find roots of an equation

Let $f(x) = x^7 + 3x^3 + 7x^2 - 1$

Say we want to solve $0 = x^7 + 3x^3 + 7x^2 - 1$.

That's HARD.

Let's estimate it.



Notice, the tangent line hits the x-axis closer to the root.

Let $x_0 = 0.5$ (this is an x-value "close" to the root)

look at tangent line at x_0.

$f'(x) = 7x^6 + 9x^2 + 14x$

$f'(1/2) = 7(\frac{1}{2})^6 + 9(\frac{1}{2})^2 + 14(\frac{1}{2})$

$= 9.3594$

Equation of tangent at x_0:

$y - 1.1328 = 9.3594(x - \frac{1}{2})$

when $y = 0$

$-1.1328 = 9.3594(x_1 - \frac{1}{2})$

$x_1 = .3790$

x_1 is closer to root than x_0.

we can do the same process with x_1 now.

$f'(.3790) = 6.619 \qquad f(.3790) = 0.170$

tangent line at x_1: $y = 6.619(x - .3790) + 0.170$

solve: $0 = 6.619(x_2 - .3790) + 0.170$

$x_2 = 0.353$

Newton's Method

Step 1: Pick a place to start on the x-axis. Call it x_0.

Step 2: Tangent line at x_0 is $y = f(x_0) + f'(x_0)(x - x_0)$

Tangent line intersects x-axis at $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

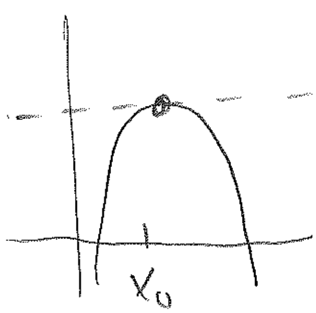
x_1 is your new value

Step 3: Repeat w/ your new x-value. In general, the "next" value is:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 4: Keep going until your x_i's stabilize. Where they stabilize is the approximation to your root!

Caution: don't pick an x_0 at a critical point ($f'(x_0) = 0$).



this tangent line never intersects x axis.

__Example:__ Approx a root of $f(x) = x^2 - x - 1$ near $x_0 = 1$

$f'(x) = 2x - 1$

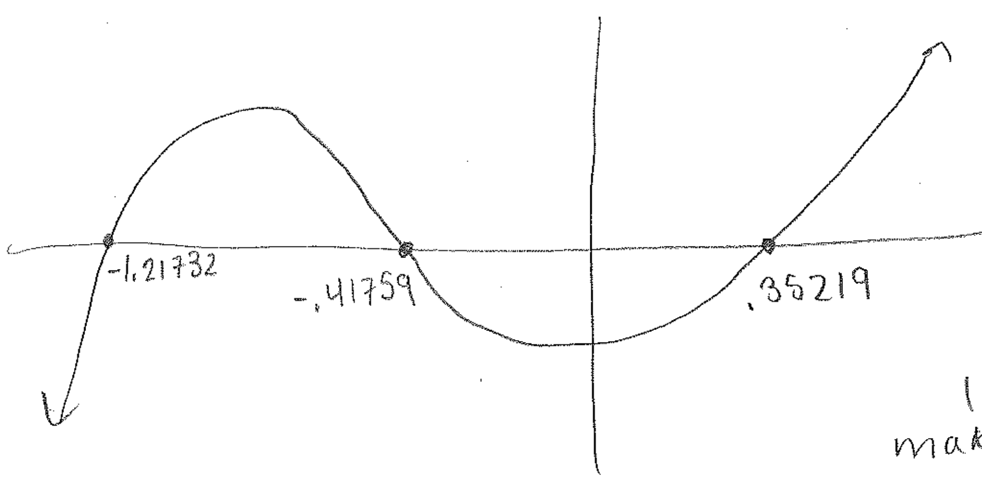
i	x_i	$f(x_i)$	$f'(x_i)$	$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
0	1	-1	1	$x_1 = 1 - \frac{-1}{1} = 2$
1	2	1	3	$x_2 = 2 - \frac{1}{3} = \frac{5}{3} \approx 1.667$
2	5/3	1/9	7/3	$x_3 = 5/3 - \frac{1/9}{7/3} = \frac{35}{21} - \frac{1}{21} = \frac{34}{21} \approx 1.619$

Know how to do this by hand

But for harder problems, use an applet:

www.math.dartmouth.edu/~klbooksite/appfolder/213unit/Newton.html

Back to $f(x) = x^7 + 3x^3 + 7x^2 - 1$



"using applet"

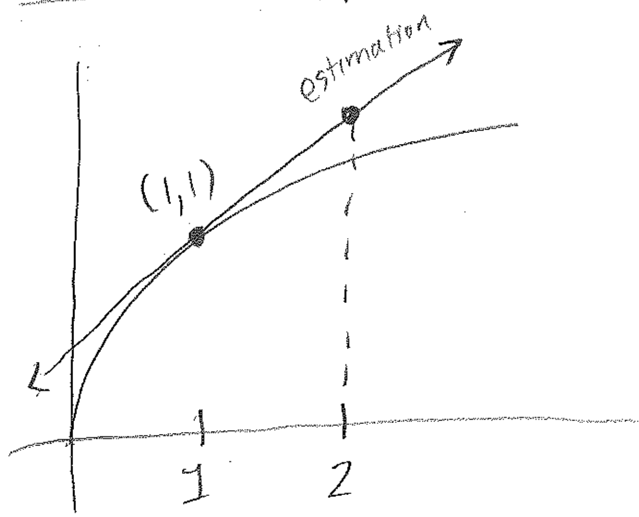
* the applet will stop iterating when it has levelled off.

In WeBWorK you may end up inputting the same #'s*

Another Application: Linearization

Linearization = Use straight lines to approximate curves.

Example: Approximate $\sqrt{2}$



Idea: we know $\sqrt{1} = 1$.
Let's use that

Use the tangent line at 1 to estimate $\sqrt{2}$.

Procedure

1) Find tangent line at 1
if $f(x) = \sqrt{x}$ then $f'(x) = 1/2\sqrt{x}$
$f'(1) = 1/2$
tangent line: $y = 1 + \frac{1}{2}(x-1)$

2) Plug in value you're interested in.
Want $\sqrt{2}$. So plug in 2
$1 + \frac{1}{2}(2-1) = 1.5 \approx \sqrt{2}$ $(\sqrt{2} = 1.414\ldots$

We have $f(x)$ diff'ble at a.

Tangent Line at a: $\underbrace{y = f(a) + f'(a)(x-a)}$ ← this isn't surprising.

this is a linear approximation of $f(x)$ when x is close to a

Call this line $L(x)$

$(L(x) = f(a) + f'(a)(x-a))$

Example: Approx $\sqrt{5}$

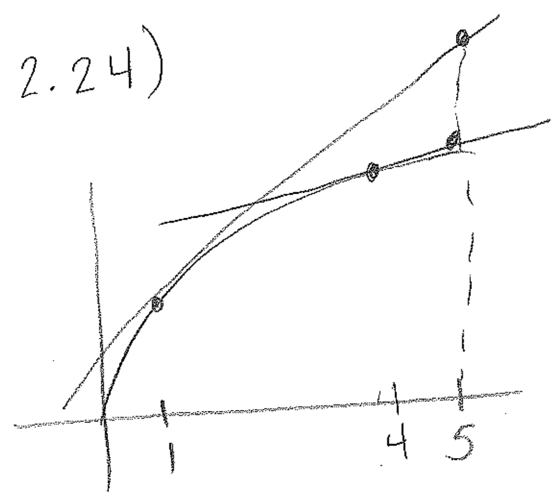
Lets use $L(x)$ from before

$L(x) = 1 + \frac{1}{2}(x-1)$

$L(5) = 1 + \frac{1}{2}(5-1) = 3$ $\quad (\sqrt{5} = 2.24)$

Not a great approx.

Need to take a tangent line closer



Find tangent line at $x=4$

$L(x) = 2 + \frac{1}{4}(x-4)$

$L(5) = 2 + \frac{1}{4}(5-4) = 2.25$ Much closer!

Interesting fact: $L(a) = f(a)$, $L'(a) = f'(a)$ (since $L(x)$ is tangent line at a)

We can design a function where second derivatives at a are also equal.

$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ $\quad (p_2''(a) = f''(a))$

or cubic $p_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{2\cdot 3}f'''(a)(x-a)^3$

and so on...